Flexion and Skewness in Map Projections of the Earth

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Abstract

Tissot indicatrices have provided visual measures of local area and isotropy distortions. Here we show how large-scale distortions of flexion (bending) and skewness (lopsidedness) can be measured. Area and isotropy distortions depend on the map-projection metric; flexion and skewness, which manifest themselves on continental scales, depend on the first derivatives of the metric. We introduce new indicatrices that show not only area and isotropy distortions but flexion and skewness as well. We present a table showing error measures for area, isotropy, flexion, skewness, distances, and boundary cuts, allowing us to compare different world-map projections. We find that the Winkel-Tripel projection (already adopted for world maps by National Geographic) has low distortion on most measures and excellent quality overall.

Keywords: maps, Earth, projection, curvature, indicatrices

Résumé

Les indicatrices de Tissot fournissent des mesures visuelles pour les distorsions des isotropies et des régions locales. Dans l'article, on montre comment on peut mesurer les distorsions à grande échelle de la flexion et de l'asymétrie. Ces distorsions dépendent des paramètres de projection cartographique: la flexion et l'asymétrie, qui se manifestent à l'échelle continentale, dépendent des premières dérivées des mesures. On présente de nouvelles indicatrices qui reflètent non seulement des distorsions de la région et de l'isotropie, mais aussi la flexion et l'asymétrie. Dans un tableau, on donne les mesures d'erreur de la région, de l'isotropie, de la flexion, de l'asymétrie, des distances et de la limite des frontières, ce qui nous permet de comparer différentes projections pour des cartes mondiales. On a découvert que la projection Winkel-Tripel (déjà adoptée pour les cartes mondiales par le National Geographic) donne une faible distorsion pour la plupart des mesures et offre une excellente qualité dans l'ensemble.

Mots clés : cartes, Terre, projection, courbure, indicatrices

1. Introduction

Tissot (1881) indicatrices have been very useful for providing a visual presentation of local distortions in map projections in a simple and compelling way. A small circle of tiny radius (say 0.1° of arc in radius) is constructed at a given location, then enlarged and projected on the map at that location. This always produces an ellipse. Usually one favours conformal map projections that minimize the changes in scale factor, equal-area projections that minimize anisotropy, or, recently, map projections that are neither conformal nor equal area but minimize both scale and isotropy errors judiciously (e.g., the Winkel-Tripel used by the National Geographic Society for world maps).

The Tissot ellipse at a given location is specified by three parameters: the major axis, the minor axis, and the orientation angle \( \theta \) of the major axis of the ellipse.
Geometrically, from differential geometry (and General Relativity), we know that the measurement of local distances is measured by the metric tensor $g_{ab}$, where $a$ and $b$ can each take the values $x$ or $y$, yielding three independent components. Locally, for two nearby points separated by infinitesimal map coordinate differences $dx$ and $dy$, the true distance between these two points on the globe is given by

$$dx^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2 \quad (1)$$

The Tissot ellipse (major axis, minor axis, and orientation angle, $\theta$) can be calculated from the three components of the metric tensor. Thus, the Tissot ellipse essentially carries the information on the metric tensor for the map. It tells us how local infinitesimal distances on the map correlate with local infinitesimal distances on the globe.

As we will see, however, the Tissot ellipse does not carry all the information related to distortions, as has been noted previously. Others (Stewart 1943; Peters 1975, 1978; Albinus 1981; Caneters 1989, 2002; Laskowski 1997a, 1997b) have noted that there are finite distortions apart from those of the Tissot ellipse.

Earlier authors have previously depicted finite size distortions by several methods, including showing faces on a globe (Reeves 1910; Gedymin 1914); $30^\circ \times 30^\circ$ equiangular quadrilaterals (Chamberlin 1915); a net of 20 spherical triangles in an icosahedral arrangement (Fisher and Miller 1944); $150^\circ$ great circle arcs; and circles of $15^\circ$ radius (Tobler 1964).

For the Oxford-Hammond Atlas of the World (1993), new conformal map projections (“Hammond Optimal Conformal Projections”) were designed for the continents. Following the Chebyshev criterion (Snyder 1993), the root-mean-square (rms) scale factor errors were minimized by producing a constant scale factor along the boundary of the continent. (For a circular region, this conformal map would be a stereographic projection.) Tailoring the boundary to the shape of each continent reduced the errors relative to those in a simple stereographic projection. In touting the advantages of their projection, the Hammond Atlas did the following experiment. Following Reeves (1910), they constructed a face on the globe with a triangular nose, a straight (geodesic) mouth, and eyes that were pairs of concentric circles on the globe. They then showed this face with various map projections. In the gnomonic, the mouth was straight but the eye circles were not circular and were not concentric (what we would call “skewness”). The Mercator projection had the mouth smiling (what we would call “flexion”), and, although the eyes were circular, they were not concentric. The Hammond Optimal Conformal projections did a bit better on these qualities because the gradients of the scale factor changes were small, so the flexion and skewness were small, although, of course, not zero.

Section 2 below introduces the concept of flexion, by which a map projection can cause artificial bending of large structures. Section 3 shows another distortion: skewness, which represents lopsidedness and an asymmetric stretching of large structures. We show a simple way to visualize these distortions in section 4, which introduces the Goldberg-Gott Indicatrices. In section 5 we derive a differential geometry approach to measuring flexion and skewness. While readers interested in computing flexion and skewness on projections not included in this paper should refer to section 5, it is highly technical, and those interested in seeing results may skip directly to section 6, in which we discuss Monte Carlo estimates of the distortions for a number of projections, a ranking of map projections, and our conclusions.

2. First Distortion: Flexion

The local effects shown by the Tissot ellipse are not the only distortions present in maps. The terms “flexion” (or bending) and “skewness” (or lopsidedness, discussed in the next section) describe curvature distortions visible on world maps; the terminology stems from a similar effect in gravitational lensing (Goldberg and Bacon 2005).

One can think about flexion in the following way. Imagine a truck going along a geodesic of the globe at unit angular speed (say one radian per day). Now imagine the image of that truck on the map, moving along. If the map were perfect – if it had zero flexion and zero skewness – then that truck would move in a straight line on the map with constant speed. Its velocity vector on the map

$$\mathbf{v} = \frac{dx}{dt} \quad (2)$$

would be a constant, where $\tau$ is the angle of arc length in radians travelled by the truck along the geodesic on the globe. Its acceleration

$$\mathbf{a} = \frac{dv}{d\tau} \quad (3)$$

would be zero. Of course, this cannot be true for a general geodesic. In the general case, the image of the truck suffers an acceleration as it moves along. The acceleration vector, $\mathbf{a}$, in the two-dimensional map has two independent components: $a_\perp$ (which is perpendicular to the truck’s velocity vector at that point), and $a_\parallel$ (which is parallel to its velocity vector at that point).

The perpendicular acceleration, $a_\perp$, causes the truck to turn without changing its speed on the map. This causes flexion, or bending, of geodesics. We define the flexion along a given geodesic at a given point to be

$$f = \frac{a_\perp}{v} \quad (4)$$
or, more usefully,

\[ f = \frac{dv_\perp}{d\tau} \]  \hspace{1cm} (5)

In this form, we may define

\[ da = \frac{dv_\perp}{v} \]  \hspace{1cm} (6)

where \( a \) is the angle of rotation suffered by the velocity vector. Remember, if \( a_\perp \) is the only acceleration present, then the velocity vector of the truck on the map does not change in magnitude but only rotates in angle. Thus \( f = da/d\tau \), represents the angular rate at which the velocity vector rotates divided by the angular rate at which the truck moves on the globe.

Skewness and flexion express themselves only at large scales. They are not noticeable at infinitesimal scales, where the metric contains all the information one needs, but become noticeable on finite scales, and their importance grows with the size of the area being examined. Flexion and skewness are thus important at continental scale and larger on a world map.

It is possible to design a map projection that has zero flexion: the gnomonic projection shows all great circles as straight lines. However, it also exhibits anisotropy, scale changes, and skewness, and, at best, it can show only one hemisphere of the globe.

2.1 EXAMPLE 1: THE STEREOGRAPHIC PROJECTION

As an example, consider a truck travelling on the equator as seen in the polar stereographic projection (see Figure 1). In the stereographic projection, the north pole is in the centre of the map and the equator is a circle around it. As the truck circles the equator (the equator is a geodesic, so the truck drives straight ahead on the globe), it travels around a circle on the map. By azimuthal symmetry, the truck circles the equatorial circle on the map at a uniform rate. The velocity vector of the truck on the map rotates a complete 360˚ (\( 2\pi \) radians) as the truck circles the equator, traversing 360˚ of arc on the globe. So the flexion is \( f = 1 \), for a point on the equator, for a geodesic pointing in the direction of the equator. The flexion is defined at a point and for a specific geodesic travelling through that point.

For an arbitrary point on the equator in the stereographic projection and a geodesic pointing in the direction of a meridian of longitude (also a geodesic), the flexion is zero, because these geodesics are shown as radial straight lines in the polar stereographic projection, and the velocity vector of the truck driving along the geodesic does not turn as it travels north.

The stereographic projection has the property that every great circle (geodesic) on the globe is shown as a circle on the map, except for a set of measure zero that pass through the north pole (i.e., the meridians of longitude).
Thus, by the argument given above, the average flexion integrated around a random great circle must be $\langle f \rangle = 1$, because the truck’s velocity vector on a random great circle must rotate by $360^\circ$ as it circles the $360^\circ$ of arc completing that great circle on the globe. The magnitude of the velocity vector of the truck on the map is larger the further from the north pole it is, and so its rotation per angle of arc of truck travel on the globe is larger there as well, and the flexion along that random geodesic is larger the further away from the pole one is; the integrated average value along the whole great circle is $\langle f \rangle = 1$.

### 2.2 EXAMPLE 2: THE MERCATOR PROJECTION

The Mercator projection (see Figure 2) is conformal, and so only the scale factor changes as a function of position on the map (i.e., $g_{xx} = g_{yy}$ and $g_{xy} = 0$, and the Tissot ellipses are all circles with radii proportional to $1/g_{xx}$). But there is bending. The northern boundary between the continental United States and Canada at the forty-ninth parallel of latitude is shown as a straight line in the Mercator map, but really it is a small circle that is concave to the north. If one drove a truck down that border from west to east, one would have to turn the steering wheel slightly to the left so that one was continually changing direction. The great circle route (the straightest route) connecting Washington state and Minnesota (both at the forty-ninth parallel) is a straight line that goes entirely through Canada. This straight line on the globe, when extended, passes south of the northern part of Maine, so that the continental United States is bent downward like a frown in the Mercator map (see Figures 3 and 4). Likewise, on a Mercator map Maine sags below the line connecting Washington state and Minnesota, while on the globe this is not true.

In the Mercator projection, flexion is zero along the equator and along all meridians of longitude, but these are a set of measure zero. A random geodesic is a great circle that is inclined at some angle between $0^\circ$ and $90^\circ$ with respect to the equator. On the Mercator map, this is a wavy line that bends downward in the northern hemisphere and, by symmetry, upward in the southern. Since the curvatures are equal and opposite in the two hemispheres, the average flexion $\langle f \rangle = 0$, but this is misleading, because the flexion at each point off the equator is not zero. So, if we are rating map projections by the amount of flexion they contain, we should use the absolute value of the flexion instead: $|f|$. In a region where the flexion does not change signs (such as the northern hemisphere in the Mercator projection or the entire stereographic map), the total bending of a geodesic segment will be the integral of the flexion $|f|$ over that segment. In fact, in section 6.1 we will evaluate the overall flexion on a map by simply picking random points on the sphere and random directions for geodesics going through them, then calculating the absolute value for the flexion for all random points on the globe and random directions through them.

### 2.3 A GLOBAL FLEXION MEASURE

We can calculate the flexion for any point in the Mercator (or any other) projection through any geodesic using

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Figure 2. The indicatrix map for a Mercator projection.
spherical trigonometry. As a reminder to the reader, the Mercator projection uses the mapping

\[
x = \lambda \\
y = \ln(\tan(\pi/4 + \phi/2))
\]

where, here and throughout, \(\lambda\) is the longitude expressed in radians and \(\phi\) is the latitude expressed in radians.

On the map, the angle rotation along the geodesic with an azimuth, \(\theta\), is calculated by constructing a thin spherical triangle with side–angle–side given by \((\pi/2 - \phi, \theta, dt)\):

\[
d\alpha = \pi - \beta - \theta
\]

because the geodesic intersects the north–south meridian (a vertical line in the Mercator map) at an angle of \(\theta\).
initially and at an angle \( \pi - \beta \) at the other end, where \( \beta \) is the angle in the spherical triangle at the other end of the \( dt \) side. Solving for \( d\theta \) using spherical trigonometry in the limit as \( dt \) goes to zero, we find that
\[
f = \sin \theta \tan \phi
\] (10)
Thus, for \( \theta = \pi/2 \), an east–west geodesic, we find that
\[
f_{\text{EW}} = \tan \phi
\] (11)
so that in the northern hemisphere, travelling east, one’s geodesic is bending clockwise with \( da/dt = \tan \phi \).
Therefore, the east–west geodesic bends downward. For, \( \theta = 0 \), a north–south geodesic, the flexion is zero, as we expect, since the meridians of longitude are straight in the Mercator map. If we average over all azimuths at a given point, we find
\[
\langle |f(\phi)| \rangle = |\tan \phi| \frac{2}{\pi}
\] (12)
Now we can integrate this over all points on the sphere to produce the average flexion over the whole sphere \( F \). Taking advantage of the symmetry between the northern and southern hemispheres, we can integrate only over the northern hemisphere (where \( dA = 2\pi \cos \phi d\phi \)), yielding
\[
F = \frac{1}{2} \int 2\pi \cos \phi \tan \phi \frac{d\phi}{\pi} = \frac{2}{\pi}
\] (13)
The flexion is less than that of the stereographic projection because of the 180˚ boundary cut along the longitude line at the international date line. A geodesic is a great circle on the globe, and if this is shown as a closed curve on the map that is always concave inward (the best possible case), it will always have a total rotation of the velocity vector of 360˚ and so will have an average integrated flexion of 1. If there is a boundary cut, the great circle does not have to close on the map (it has two loose ends at the boundary cut) and so need not completely rotate by 360˚.

3. Second Distortion: Skewness

Acceleration in the direction parallel to the velocity vector of the truck \( a_{||} \) causes the truck to increase its speed along the geodesic curve without causing any rotation. This causes skewness, because, as the truck accelerates, it covers more distance on the map on one side of a point than on the other, so the point in question will not be at the centre of the line segment of arc centred on that point on the sphere.

Consider a segment of a meridian of longitude on the globe centred at 45˚ north latitude. Going from south to north along that geodesic in the Mercator map, the truck is accelerating with \( a_{||} = 0 \); the scale factor is getting larger and larger the further north one goes, so, as the truck continues to cover equal arc length on the globe, it covers larger and larger distances on the map. Thus, the centre of the segment (at 45˚N latitude) is not centred on the segment on the map.

We define the skewness
\[
s = \frac{a_{||}}{v}
\] (14)
Taking the explicit case of the Mercator projection, we find that
\[
v_y = \frac{dy}{d\phi} = \frac{1}{\cos \phi}
\] (15)
and thus
\[
a_{||} = \frac{\tan \phi}{\cos \phi}
\] (16)
so the skewness (for a vector pointed N–S) is simply
\[
s = \tan \phi
\] (17)
The skewness at 45˚ is 1, showing a lopsidedness toward the north. Given this relation, the skewness is positive (northward lopsidedness) in the northern hemisphere and negative (southward lopsidedness) in the southern hemisphere.

Consider a geodesic through a point in the northern hemisphere tipped at an azimuth angle of \( \theta \) with respect to north. The only thing increasing the speed of the truck is the gradient of the scale factor as one moves northward, so the amplitude of the parallel acceleration is equal to the maximum acceleration (obtained going straight north) times \( \cos \theta \). To get the average of the absolute value of the skewness for all geodesics through that point at all random angles \( \theta \), one simply integrates over \( \theta \):
\[
\langle |s| \rangle = \langle |\tan \phi| \rangle \frac{\int_0^{\pi/2} \cos \theta d\theta}{\pi/2} = \langle |\tan \phi| \rangle \frac{2}{\pi}
\] (18)
As with the flexion, we can integrate this over all points on the sphere to produce the average skewness over the whole sphere \( S \). Similarly, we find
\[
S = \frac{2}{\pi}
\] (19)
Notice that this is exactly the same value as the average flexion, \( F \), for the Mercator map. We will find that for conformal projections, the average absolute values of
skewness and flexion at a given point and over the whole
globe are always equal. (This is true only for conformal
projections; for general projections, the skewness and
flexion can be different, as illustrated by the gnomonic
projection, which has zero flexion but non-zero
skewness.)

In the Mercator projection the skewness is zero at the
equator, as we would expect from symmetry considera-
tions. Likewise, the flexion is zero for any geodesic line
evaluated at a point on the equator. So the Mercator map
has perfect local shapes along the equator, uniform scale
along the equator (Tissot ellipses are all equal-size circles),
and zero flexion and skewness along geodesics in any
direction from points on the equator.

While much of the analysis in this work specifically
addresses distortions in maps of the earth, these effects
must be taken into account in other maps as well. One of
the authors has recently produced a conformal "map of
the universe" (Gott and others 2005) based on the
logarithm map of the complex plane. The horizontal
coordinate is the celestial longitude in radians, yielding a
360˚ panorama from left to right. The vertical coordinate
is

\[ y = \ln(d/r_0) \]  

(20)

where \( d \) is the distance and \( r_0 \) is the radius of the Earth.
The distance scale goes inversely as the distance, allowing
us to plot everything from satellites in low Earth orbit, to
stars and galaxies, to the cosmic microwave background
on one map. The map is conformal, having perfect local
shapes, but it does have flexion and skewness. Circles of
constant radius from the Earth are bent into straight lines,
for example, and a rocket travelling out from the Earth at
constant speed would be slowing down on the map.

In Figure 3, note that the continental United States also
appears lopsided on the Mercator map. The geographical
centre of the continental United States (which is in
Kansas) appears in the lower half of the country on the
Mercator map, because the scale factor on the map gets
larger and larger the further north one goes on the map.
Thus, the continental United States is lopsided toward the
north. Flexion or bending is manifest on the map as a
bending of geodesics, and skewness or lopsidedness is
manifest in the fact that the midpoint of a geodesic line
segment is not at the midpoint of that geodesic arc as
shown on the map.

4. Goldberg-Gott Indicatrices

We began this discussion with the virtues of the Tissot
indicatrices. Likewise, we have produced a simple
indicatrix to show the flexion and skewness in a map as
well as the isotropy and area properties indicated by the
Tissot indicatrices. We will refer to these as the Goldberg-
Gott indicatrices. They are constructed as follows. At a
specific point on the map, draw a circle on the globe of
radius 12˚, then plot it on the map. Inside this circle, plot
the north–south and east–west geodesics through the
central point on the map. This leaves a @ symbol on the
map. If the map were perfect, this would be a perfect circle
and the cross-arms perfectly straight, intersecting at the
centre of the circle.

We have produced such a Goldberg-Gott indicatrix,
located at the geographic centre of the continental United
States, for a Mercator projection in Figure 3.

We note that our technique is similar to that of Waldo
Tobler (1964), who projected circles of 15˚ radius on to
maps. However, it differs in that Tobler’s circles do
not have their centres marked, nor are perpendicular
geodesics drawn from their centres, as in our indicatrices,
and so they convey no information on flexion and
skewness. They do show the finite shape distortions of the
circles themselves, which our indicatrices also cover.
Tobler includes tables showing, for particular locations on
the sinusoidal projection, the maximum and minimum
scale radius and maximum difference of radial directions
for circles of various radii on the sphere (addressing the
Tissot issues of size and isotropy on finite circles but
not flexion and skewness). Tobler also calculated errors
in miles and angular errors for 300 random triangles
within spherical circles of various radii in different
map projections and considered area errors for such
random triangles within land areas in various world map
projections.

Using our indicatrices, one can see in Figure 3 that the
north–south geodesic is straight but the east–west
geodesic is bent downward. This shows dramatically the
flexion in this region of the Mercator map. One can even
read off the average value of the flexion by hand: take
a protractor and measure the tangent to the east–west
geodesic at the two ends of the cross bar; measure the
difference in the angle orientation of the two (this gives
the integrated flexion along 24˚ of the globe); then divide
that angle difference by 24˚, which yields the average value
of the flexion along that curve.

The skewness is also visible in that the centre of the cross
is below the centre of the circle, showing the lopsidedness
to the north. In fact, one can observe the skewness in any
direction from the centre by seeing how far the centre
of the cross is from the centre of the circle in different
directions. For comparison, Figure 4 shows the con-
tinental United States in an oblique Mercator projection,
in which the east–west geodesic through the geographic
centre of the continental United States is now the equator
of the Mercator projection. The flexion and skewness
along the equator of a Mercator map are indeed zero, so
the arms of the cross are now straight, and the circle is
now nearly a perfect circle centred on the centre of the
cross. This gives a “straight on” view of the continental
United States that more accurately portrays its appearance on the globe.

One can place the Goldberg-Gott indicatrices every 60° in longitude and every 30° in latitude on the globe to show how flexion and skewness vary over the map. Figures 5–27 provide Goldberg-Gott indicatrix maps for a number of well-known projections.

In fact, the Goldberg-Gott indicatrices can almost simply replace the Tissot indicatrices, because the shape and size of the oval in the Goldberg-Gott indicatrix is approximately the size and shape of the Tissot ellipse. The Tissot ellipse shows the (magnified) shape and size of an infinitesimal circle on the globe; the oval in the Goldberg-Gott indicatrix shows the shape and size of a finite circle (radius 12°) on the map itself at correct scale. Thus, if the map is equal area, the Goldberg-Gott indicatrices will all have equal area on the map. If the map is conformal, the Goldberg-Gott indicatrices will all be nearly perfectly circular. If there is a 2:1 anisotropy in the Tissot ellipses in a given region, the Goldberg-Gott indicatrix ovals will have that same 2:1 axis ratio.

5. A Differential Geometry Approach

Thus far, we have defined the general properties of skewness and flexion, given a few analytical results for particular map projections, and offered a graphic approach for describing and interpreting flexion and skewness on maps. In this section, we approach the matter somewhat differently, producing general analytical results for all projections as well as a prescription for measuring the flexion and skewness analytically.

5.1 Coordinate Transforms

Let us consider a spherical globe with coordinates

\[ x^i = \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \]  

(21)

Note that, here and throughout, we will use \( x^i \) to refer to coordinates in the globe frame, and \( x^i' \) to refer to coordinates in the map frame.

Figure 5. The indicatrix map for an azimuthal equidistant projection.

Figure 6. The indicatrix map for a Briesemeister projection.
Figure 7. The indicatrix map for an Eckert IV projection.

Figure 8. The indicatrix map for an Eckert VI projection.

Figure 9. The indicatrix map for a Gall-Peters projection.
On the globe, the metric is
\[ g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 \phi \end{pmatrix} \] (22)
such that, as always, the distance between two points can be expressed as
\[ dl^2 = dx^a dx^b g_{ab} \] (23)
Now, consider an arbitrary 2D coordinate transformation:
\[ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^1(\phi, \lambda) \\ x^2(\phi, \lambda) \end{pmatrix} \] (24)
Of course, from this definition, we may easily compute a local transformation matrix:
\[ \Lambda^a_{\tilde{a}} = \frac{\partial x^a}{\partial \tilde{x}^\tilde{a}} \] (25)
The inverse matrix is \( \Lambda_{\tilde{a}}^a = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial x^a} \). The metric in the map frame is
\[ g_{\tilde{a}\tilde{b}} = \Lambda^c_{\tilde{a}} \Lambda^d_{\tilde{b}} g_{cd} \] (26)
From this we may then compute the Christoffel symbols of general relativity:
\[ \Gamma^a_{bc} = \frac{1}{2} g^{ae} (\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{bc}) \] (27)
where standard convention tells us to sum over identical indices in the upper and lower positions and where a comma indicates a partial derivative with respect to a coordinate.

In practice, actually computing the Christoffel symbols for an arbitrary projection is not simple. To do this analytically, we must have an analytical form for the map inversion. However, we make available a numerical

Figure 12. The indicatrix map for a gnomonic cube projection.

Figure 13. The indicatrix map for a Gott-Mugnolo projection.
code to compute the Christoffel symbols for all map projections discussed in this work on our projections Web site (see below).

5.2 ANALYTICAL FORMS OF FLEXION AND SKEWNESS

The whole point of computing the Christoffel symbols is that we want to address a very simple question: How are large structures distorted when projected onto a map? Clearly, to an observer on the globe, a straight line is easy to generate. Point in a particular direction and start driving (assuming your car can drive on the ocean) with the steering wheel set straight ahead. Drive for a fixed distance in units of angles or radians. Record all points along the way.

Geometers, of course, know this route as a geodesic, and, if we consider $\tau$ to represent a physical distance on the

![Figure 14. The indicatrix map for a Gott-Mugnolo elliptical projection.](image1.png)

![Figure 15. The indicatrix map for a Gott equal-area elliptical projection.](image2.png)
surface of the Earth, then the geodesic equation may be expressed as follows:

\[
\frac{du^e}{d\tau} = -\Gamma^e_{bc} u^b u^c
\]

where

\[
u^e = \frac{dx^e}{d\tau}
\]

Equation (28) describes the bending of straight lines on a particular map projection; thus, if all of the Christoffel symbols could vanish, we would clearly have a perfect Cartesian map. Not possible, of course.

But what is the physical significance of the Christoffel symbols? Since the lower indices are symmetrical by inspection, there are six unique symbols. What do they mean?

5.2.1 Analytical Flexion

We define a vector oriented in a particular direction:

\[
\tilde{u}^e(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}
\]

In reality, this is not the unit vector, since

\[
|\tilde{u}|^2 = g_{ab} \tilde{u}^a \tilde{u}^b \neq 1
\]

Of course, we could define a true unit vector:

\[
u(\theta) = l(\theta) \tilde{u}(\theta)
\]
where \( l \) is the “length” in grid coordinates of the unit vector. This has a value of

\[
l(\theta) = \frac{1}{\sqrt{\cos^2 \theta_{g11} + \sin^2 \theta_{g22} + 2 \sin \theta \cos \theta_{g12}}} \quad (33)
\]

Of course, it is clear that at any point on the map, the set of all \( u(\theta) \) represents an ellipse – the Tissot ellipse.

We define the flexion in the following manner: follow a particular geodesic a distance, \( dt \) (measured in, for
example, radians). On the map, the geodesic will change direction by an angle $\theta$. The ratio

$$f = \frac{d\theta}{d\tau}$$

is the flexion. Note that for all polar projections, the equator will have a flexion of 1.

In terms of the geodesic equation, the flexion can be expressed as

$$f(\theta) = l(\theta) \left( \frac{d\vec{u}}{d\tau} \times \vec{u} \right) = l(\theta)(\Gamma^1_{ab} \vec{u}^a \vec{u}^b \vec{u}^1 - \Gamma^1_{ab} \vec{u}^a \vec{u}^b \vec{u}^2)$$

where $\vec{u}$ denotes a vector.

Some light can be shed on equation (35) if we expand the expression explicitly into trigonometric functions:

$$s(\theta) = l(\theta) \left( \frac{d\vec{u}}{d\tau} \cdot \vec{u} \right) = l(\theta)(\Gamma^1_{ab} \vec{u}^a \vec{u}^b \vec{u}^1 + \Gamma^2_{ab} \vec{u}^a \vec{u}^b \vec{u}^2)$$

As with the flexion, we may expand these out explicitly:

$$s(\theta) = l(\theta) \left[ \Gamma^1_{11} \cos^3 \theta + (2\Gamma^1_{12} + \Gamma^1_{11}) \cos^2 \theta \sin \theta \right]_{(2\Gamma^2_{12} + \Gamma^1_{22}) \sin \theta \cos^2 \theta + \Gamma^2_{22} \sin^3 \theta}$$

5.2.2 Analytical Skewness

Essentially, skewness means, for example, that if you walk (initially) north for 1000 miles, or walk south for 1000 miles, you will cover different amounts of map coordinate.

The skewness along a geodesic can be defined similarly to the flexion:

$$s(\theta) = l(\theta)\left( \frac{d\vec{u}}{d\tau} \times \vec{u} \right)$$

Figure 20. The indicatrix map for a Lambert azimuthal projection.
A map with no skewness will have
\[
\begin{align*}
\Gamma_{11}^1 &= \Gamma_{22}^1 = 0 \\
\Gamma_{11}^2 &= -2\Gamma_{12}^1 \\
\Gamma_{22}^2 &= -2\Gamma_{12}^2
\end{align*}
\tag{40}
\]
We know of no projections with zero skewness everywhere.

5.3 PROJECTIONS WITH STRAIGHTFORWARD ANALYTICAL RESULTS

5.3.1 The Gnomonic Projection
The gnomonic projection (see Figure 12) is particularly interesting. It has a coordinate transformation
\[
\begin{pmatrix} x \\ y \end{pmatrix}_g = \begin{pmatrix} \cot \phi \cos \lambda \\ \cot \phi \sin \lambda \end{pmatrix}
\tag{41}
\]
that is directly invertible to yield
\[
\begin{pmatrix} \phi \\ \lambda \end{pmatrix} = \begin{pmatrix} \cos^{-1} \sqrt{x^2 + y^2} \\ \tan^{-1} \left( \frac{y}{x} \right) \end{pmatrix}
\tag{42}
\]
The coordinate transformation is thus
\[
\Lambda^g = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}
\tag{43}
\]
We thus have the map metric
\[
\begin{pmatrix} g_{11} \\ g_{12} \\ g_{21} \\ g_{22} \end{pmatrix} = \begin{pmatrix} \frac{1 + y^2}{(1 + x^2 + y^2)} & \frac{-xy}{(1 + x^2 + y^2)} \\ \frac{-xy}{(1 + x^2 + y^2)} & \frac{1 + x^2}{(1 + x^2 + y^2)} \end{pmatrix}
\tag{44}
\]
We can compute the Christoffel symbols in the normal way. We find that
\[
\begin{align*}
\Gamma^1_{11} &= -\frac{2x}{1 + x^2 + y^2} \\
\Gamma^2_{11} &= 0 \\
\Gamma^1_{12} &= -\frac{y}{1 + x^2 + y^2} \\
\Gamma^2_{12} &= -\frac{x}{1 + x^2 + y^2} \\
\Gamma^1_{22} &= 0 \\
\Gamma^2_{22} &= -\frac{2y}{1 + x^2 + y^2}
\end{align*}
\]
This clearly satisfies the requirements of equation (37), but not those of equation (40); thus, the gnomonic projection produces straight but skewed geodesics.

5.3.2 The Stereographic Projection
The stereographic projection (see Figure 1) is conformal, and, thus, all the Tissot ellipses are circles. Does this mean that there is no skewness in this projection? No, as we have already seen. The stereographic projection has the coordinate transformation
\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
\tan(\pi/4 + \phi/2) \cos \lambda \\
\tan(\pi/4 + \phi/2) \sin \lambda
\end{pmatrix}
\]

Figure 23. The indicatrix map for a Mollweide projection.

Figure 24. The indicatrix map for a polyconic projection.
which, again, is directly invertible to yield
\[
\begin{pmatrix} \phi \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} - 2 \tan^{-1} \left( \sqrt{x^2 + y^2} \right) \\ \tan^{-1} \left( \frac{\pi}{2} \right) \end{pmatrix}
\]  
(47)

The coordinate transformation is
\[
\Lambda^a_i = \begin{pmatrix} \frac{2x}{\sqrt{x^2 + y^2} (1 + x^2 + y^2)} & \frac{2y}{\sqrt{x^2 + y^2} (1 + x^2 + y^2)} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{pmatrix}
\]  
(48)

This can be used to compute the metric on the map,
\[
g_{ab} = \begin{pmatrix} \frac{4}{(1 + x^2 + y^2)^2} & 0 \\ 0 & \frac{4}{(1 + x^2 + y^2)^2} \end{pmatrix}
\]  
(49)

which clearly indicates that all Tissot ellipses will be circular.

From these, of course, we can compute the Christoffel symbols:
\[
\Gamma^1_{22} = -\Gamma^1_{12} = -\Gamma^2_{12} = \frac{2x}{1 + x^2 + y^2}
\]  
(50)

\[
\Gamma^2_{11} = -\Gamma^2_{22} = -\Gamma^1_{12} = \frac{2y}{1 + x^2 + y^2}
\]  
(51)

It is clear from inspection that geodesics are generally neither straight nor unskewed.

Moreover, it is clear that \( l(\theta) \) is independent of orientation, since the map is conformal. In general, it can be shown to be
\[
l = \frac{1}{2} \left( 1 + x^2 + y^2 \right)
\]  
(52)
In the stereographic projection, a circle of radius 12° on the globe is a perfect circle on the map, but the centre of the circle on the globe is not at the centre of the circle on the map (see Figure 1), and thus there is skewness.

### 6. Discussion

#### 6.1 NUMERICAL ANALYSIS OF STANDARD MAP PROJECTIONS

Not all projections produce such simple results. Thus, in general, we will want to compute the local flexion and skewness numerically. Our approach is as follows.

For each projection we chose 30,000 points selected randomly on the surface of a globe. For each of these points, we chose a random direction to start a geodesic. We follow that geodesic using small steps ($d \theta \approx 10^{-5}$ rad) numerically and use standard difference methods to compute the map velocity and acceleration along the geodesic. We are thus able to compute the metric and the Christoffel symbols (and thus the flexion and skewness) directly. Our Interactive Data Language (IDL) code is available to the interested reader on our projection Web page (see below). Likewise, we also do a distance test, in which pairs of points $(i, j)$ are chosen at random and the distance is measured both on the globe and on the map. This is a somewhat different perspective from simply inspecting the Goldberg-Gott indicatrices at a few locations, since we are now doing a uniform sampling over the surface of the globe rather than a uniform sampling over the map. When looking at the indicatrix map we can occasionally get a distorted view as to the quality of a particular projection. Some (like the Mercator) have relatively good fits over most of the globe, but the high latitudes can, in principle, be projected to infinite areas, and, thus, the reader may erroneously think the Mercator infinitely bad. By sampling uniformly over the globe, we get a fair assessment of the overall quality of a particular projection.

We define a number of fit parameters: $I$, corresponding to errors in the local isotropy (zero for conformal projections); $A$, corresponding to errors in the area (zero for equal-area projections); $F$, corresponding to flexion (defined in the discussion of flexion, above); $S$, corresponding to skewness (also defined above); $D$, corresponding to distance errors; and $B$, corresponding to the average number of map boundary cuts crossed by the shortest geodesic connecting a random pair of points:

\[
I = \text{RMS} \left( \ln \frac{a_i}{b_i} \right) 
\]

\[
A = \text{RMS} (\ln a_i b_i - (\ln a_i b_i))
\]

\[
F = \left( \frac{1}{n} \sum |f_i| \right)
\]

\[
S = \left( \frac{1}{n} \sum |s_i| \right)
\]

\[
D = \text{RMS} \left( \ln \frac{d_{i,j,\text{map}}}{d_{i,j,\text{globe}}} \right)
\]

\[
B = \frac{L_{B}}{4\pi}
\]

where $a_i$ and $b_i$ are the major and minor axes of the local Tissot ellipses of random point $i$, $(X_i)$indicates the mean of property $X$, and $L_{B}$ is the total length of the boundary cuts.

Logarithmic errors have been used before by Vasily Kavrayskiy (1958). Lev Bugayevskiy and John Snyder (1995) adopted schemes that weighted isotropy errors, $(a_i b_i - 1)^2$, and area errors, $(a_i b_i - 1)^2$, according to one’s interest in isotropy and area. George Airy (1861) had simply given these terms equal weight.

---

Figure 27. The indicatrix map for a Winkel-Tripel (*Times Atlas*) projection.
Table 1 compares these measures for a number of standard projections, which, for fairness of comparison, we divide into several categories. The Gott elliptical, Gott-Mugnolo elliptical, and Gott-Mugnolo azimuthal projections are discussed elsewhere (Gott, Mugnolo, and Colley 2007; Gott and others 2007), where they have been applied to the Earth, Mars, the Moon, and the cosmic microwave all-sky map.

First, we show projections that represent the complete globe without interrupts. These projections are azimuthal, and the average flexion over these maps is 1.

Second, we show the set of whole-Earth projections with one 180° interrupt, including rectangular and elliptical projections. Among the complete projections with no interrupts or one 180° interrupt, there are several “winners” with respect to performance for flexion and skewness. The Lagrange projection (Figure 19) has the smallest flexion; the Winkel-Tripel (Figure 26) has the smallest skewness. For all conformal projections, the skewness is equal to the flexion.

Of all of the whole-Earth projections, the most accurate for distance measure between points is the Gott-Mugnolo azimuthal (Figure 13), followed very closely by the Lambert azimuthal (Figure 20; see Gott, Mugnolo, and Colley 2007).

The third and fourth groups show two-hemisphere and other multiple-cut projections, respectively.

In the final group, we have a projection with multiple interrupts, the gnomonic cube, which is defined piece-meal. This is a particularly interesting projection, since the gnomonic is locally flexion free, but it is clear that geodesics will not trace out straight lines in the gnomonic cube map, because they bend when they cross an edge between faces. The gnomonic cube is presented as a cross, so five edges are included in the map proper.

Table 1. The errors in isotropy, area, flexion, skewness, distances, and boundary cuts for some standard projections

<table>
<thead>
<tr>
<th>Projection</th>
<th>( I )</th>
<th>( A )</th>
<th>( F )</th>
<th>( S )</th>
<th>( D )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Azimuthal equidistant (Figure 5)</td>
<td>0.87</td>
<td>0.60</td>
<td>1.0</td>
<td>0.57</td>
<td>0.356</td>
<td>0</td>
</tr>
<tr>
<td>Gott-Mugnolo azimuthal (Figure 13)</td>
<td>1.2</td>
<td>0.20</td>
<td>1.0</td>
<td>0.59</td>
<td>0.341</td>
<td>0</td>
</tr>
<tr>
<td>Lambert azimuthal (Figure 20)</td>
<td>1.4</td>
<td>0</td>
<td>1.0</td>
<td>2.1</td>
<td>0.343</td>
<td>0</td>
</tr>
<tr>
<td>Stereographic (Figure 1)</td>
<td>0</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.714</td>
<td>0</td>
</tr>
<tr>
<td>One 180° boundary cut</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Briesemeister (Figure 6)</td>
<td>0.79</td>
<td>0</td>
<td>0.81</td>
<td>0.42</td>
<td>0.372</td>
<td>0.25</td>
</tr>
<tr>
<td>Eckert IV (Figure 7)</td>
<td>0.70</td>
<td>0</td>
<td>0.75</td>
<td>0.55</td>
<td>0.390</td>
<td>0.25</td>
</tr>
<tr>
<td>Eckert VI (Figure 8)</td>
<td>0.73</td>
<td>0</td>
<td>0.82</td>
<td>0.61</td>
<td>0.385</td>
<td>0.25</td>
</tr>
<tr>
<td>Equirectangular (Figure 10)</td>
<td>0.51</td>
<td>0.41</td>
<td>0.64</td>
<td>0.60</td>
<td>0.449</td>
<td>0.25</td>
</tr>
<tr>
<td>Gall-Peters (Figure 9)</td>
<td>0.82</td>
<td>0</td>
<td>0.76</td>
<td>0.69</td>
<td>0.390</td>
<td>0.25</td>
</tr>
<tr>
<td>Gall stereographic (Figure 11)</td>
<td>0.28</td>
<td>0.54</td>
<td>0.67</td>
<td>0.52</td>
<td>0.420</td>
<td>0.25</td>
</tr>
<tr>
<td>Gott elliptical (Figure 15)</td>
<td>0.86</td>
<td>0</td>
<td>0.85</td>
<td>0.44</td>
<td>0.365</td>
<td>0.25</td>
</tr>
<tr>
<td>Gott-Mugnolo elliptical (Figure 14)</td>
<td>0.90</td>
<td>0</td>
<td>0.82</td>
<td>0.43</td>
<td>0.348</td>
<td>0.25</td>
</tr>
<tr>
<td>Hammer (Figure 16)</td>
<td>0.81</td>
<td>0</td>
<td>0.82</td>
<td>0.46</td>
<td>0.388</td>
<td>0.25</td>
</tr>
<tr>
<td>Hammer-Wagner (Figure 17)</td>
<td>0.687</td>
<td>0</td>
<td>0.789</td>
<td>0.518</td>
<td>0.377</td>
<td>0.25</td>
</tr>
<tr>
<td>Kavrayskiy VII (Figure 18)</td>
<td>0.45</td>
<td>0.31</td>
<td>0.69</td>
<td>0.41</td>
<td>0.405</td>
<td>0.25</td>
</tr>
<tr>
<td>Lagrange (Figure 19)</td>
<td>0</td>
<td>0.73</td>
<td>0.53</td>
<td>0.53</td>
<td>0.432</td>
<td>0.25</td>
</tr>
<tr>
<td>Lambert conic (Figure 21)</td>
<td>0</td>
<td>1.0</td>
<td>0.67</td>
<td>0.67</td>
<td>0.460</td>
<td>0.25</td>
</tr>
<tr>
<td>Mercator (Figure 2)</td>
<td>0</td>
<td>0.84</td>
<td>0.64</td>
<td>0.64</td>
<td>0.440</td>
<td>0.25</td>
</tr>
<tr>
<td>Miller (Figure 22)</td>
<td>0.25</td>
<td>0.61</td>
<td>0.62</td>
<td>0.60</td>
<td>0.439</td>
<td>0.25</td>
</tr>
<tr>
<td>Mollweide (Figure 23)</td>
<td>0.76</td>
<td>0</td>
<td>0.81</td>
<td>0.54</td>
<td>0.390</td>
<td>0.25</td>
</tr>
<tr>
<td>Polyconic (Figure 24)</td>
<td>0.79</td>
<td>0.49</td>
<td>0.92</td>
<td>0.44</td>
<td>0.364</td>
<td>0.25</td>
</tr>
<tr>
<td>Sinusoidal (Figure 25)</td>
<td>0.94</td>
<td>0</td>
<td>0.84</td>
<td>0.68</td>
<td>0.407</td>
<td>0.25</td>
</tr>
<tr>
<td>Winkel-Tripel (Figure 26)</td>
<td>0.49</td>
<td>0.22</td>
<td>0.74</td>
<td>0.34</td>
<td>0.374</td>
<td>0.25</td>
</tr>
<tr>
<td>Winkel-Tripel (Times Atlas) (Figure 27)</td>
<td>0.48</td>
<td>0.24</td>
<td>0.71</td>
<td>0.373</td>
<td>0.39</td>
<td>0.25</td>
</tr>
<tr>
<td>One 360° boundary cut</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lambert azimuthal (2 hemispheres)</td>
<td>0.36</td>
<td>0</td>
<td>0.52</td>
<td>0.11</td>
<td>0.432</td>
<td>0.5</td>
</tr>
<tr>
<td>Stereographic (2 hemispheres)</td>
<td>0</td>
<td>0.39</td>
<td>0.37</td>
<td>0.37</td>
<td>0.692</td>
<td>0.5</td>
</tr>
<tr>
<td>Multiple boundary cuts</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gnomonic cube (Figure 12)</td>
<td>0.22</td>
<td>0.37</td>
<td>0.12</td>
<td>0.87</td>
<td>0.43</td>
<td>0.686</td>
</tr>
</tbody>
</table>
Geodesics bend when they cross an edge in this laid-out-cross configuration.

The above comparisons do not depend on how important each of the criteria is (i.e., what weighting is given to each measure). Peter Laskowski (1997a, 1997b) has suggested a means of ranking very disparate maps. Though his weighting scheme is not unique, as a simple illustration of how this can be done, we will minimize the sum of the squares of all six parameters, normalized to their values in the equirectangular projection:

\[
S_e = \left( \frac{I}{N_I} \right)^2 + \left( \frac{A}{N_A} \right)^2 + \left( \frac{F}{N_F} \right)^2 + \left( \frac{S}{N_S} \right)^2 + \left( \frac{D}{N_D} \right)^2 + \left( \frac{B}{N_B} \right)^2
\]

Following Laskowski (1997a, 1997b), we set the normalization constants equal to the values of these errors in the equirectangular projection \((x = \lambda, y = \varphi)\): \(N_I = 0.51, N_A = 0.41, N_F = 0.64, N_S = 0.60, N_D = 0.449, N_B = 0.25\).

The projections with the lowest values of \(S_e\) are

1. Winkel-Tripel (4.5629)
2. Winkel-Tripel (Times Atlas) (4.5687)
3. Kavrayskiy VII (4.8390)
4. Gall stereographic (5.7582)
5. Hammer-Wagner (5.7847)
6. Eckert IV (5.8519)

This approach is certainly not unique. One may take issue with the weighting of the individual parameters, with the domain over which they are applied (the whole globe as opposed to continents only, for example), or even with how the parameters are computed; we present it as a simple example of how our results may be combined with previous studies of map projections. It is interesting that the “best” map, as selected by this criterion, is the one already used by the National Geographic Society for its whole-world projection. It is also interesting to note that the Winkel-Tripel has especially low skewness.

Another nice property of this weighting scheme is that it gives a low score to pathological projections. For example, a series of \(n\) gores (made using the polyconic projection) arranged in a sunflower pattern would approximate the azimuthal equidistant projection in distance errors as \(n\) became large but would have arbitrarily low values of \(I, A, F,\) and \(S\). But if a boundary-cut term \(B\) is included, this term would blow up and save us from choosing the bad subdivided map as better than the more visually pleasing projections described above. Stretching individual pixels at the edges of the gores to fill the gaps between them would eliminate the boundary cuts; however, it would also cause the skewness to blow up, again preventing us from giving a good score to a bad projection.

Interested readers may visit our Web site at <www.physics.drexel.edu/~goldberg/projections/> to download a free IDL code to measure the flexion, area, and other measures discussed here. We have not done calculations for all known projections, but we have covered those that have available mathematical formulas and that we thought likely to do well.

### 6.2 Conclusions

We have developed two new measures of curvature distortions found in maps of the Earth. Skewness and flexion can be used to identify particularly warped sections of a map or to identify features (such as the US–Canada border) that appear as straight lines on some map projections but do not, in fact, follow geodesics. We have developed a new graphical tool, called the Goldberg-Gott indicatrix, which can be used to show these distortions, along with those in area and shape, at many points in a map and have produced indicatrix maps for a number of popular projections. Finally, we have used these measures to produce global distortion measures for different projections. We have found that the Winkel-Tripel produces low distortion on most measures and, in particular, has the lowest skewness of all projections in our sample with a 180° boundary cut.

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